GCE: Analysis, measure theory, Lebesgue integration
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Exercise 1:

Use the Fubini theorem to prove that

$$\int_{\mathbb{R}^n} e^{-|\mathbf{x}|^2} d\mathbf{x} = \pi^{n/2}$$

Here $\mathbf{x} = (x_1, x_2, ..., x_n)$. Hint: For $n = 2$, use polar coordinates.

Exercise 2:

Let $(X, \mathcal{A}, \mu)$ be a measure space, and f be in $L^1(X)$. Let for all positive integers n set $B_n = \{x \in X : n - 1 \leq |f(x)| < n\}$.

(i). Show that $\mu(B_n) < \infty$ for all $n \geq 2$.

(ii). Show that $\sum_{n=2}^{\infty} n\mu(B_n) < \infty$.

(iii). Define $C_n = \{x \in X : n - 1 \leq |f(x)| \leq n\}$. Is the sum $\sum_{n=2}^{\infty} n\mu(C_n)$ finite?

(iv). Show that $\sum_{n=2}^{\infty} \sum_{m=2}^{n} \frac{m^2}{n^2} \mu(B_m) < \infty$.

(v). Show that for $n \geq 2$

$$\int |f|^2 1_{\{|f|<n\}} = \int |f|^2 1_{\{|f|<1\}} + \sum_{m=2}^{n} \int |f|^2 1_{B_m}$$

and infer that $\sum_{n=1}^{\infty} \frac{1}{n^2} \int |f|^2 1_{\{|f|<n\}} < \infty$.

Exercise 3:

Prove or Disprove: Suppose that $f, g : \mathbb{R} \to \mathbb{R}$, with f being a measurable function, and g being a continuous function. Then $f \circ g$ is measurable. By definition, $(f \circ g)(x) := f(g(x))$, that is, it is the composition of the two functions.